Roadef'2021-Mulhouse
Decomposition of graphs and composition of polytopes
Introduction
Compositions for the stable set polytope
Other compositions
Compositions for the Weakly Connected Independent set polytope
Conclusion

Decomposition of graphs and Composition of polytopes.

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ROADEF'2021 MULHOUSE

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Outline

1 Introduction

2 Compositions for the stable set polytope

3 Other compositions

4 Compositions for the Weakly Connected Independent set polytope

5 Conclusion

Many optimization problems (Maximum stable set, maximum cut, minimum dominating set...) are NP−difficult.

In some classes of graphs obtained by

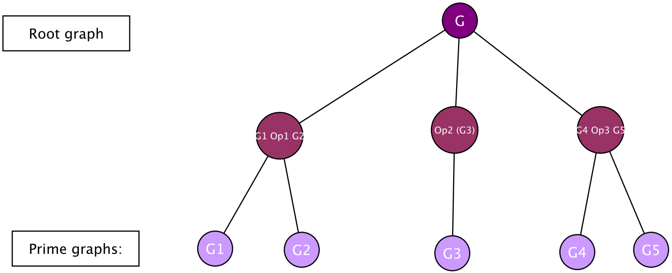
- *excluding some subgraphs*
- *decomposition and composition*

These problems are easier to solve.

A family of decomposable graphs uses

- a set of prime graphs
- a set of operations

- Trees (Acyclic and connected)
  prime graph: G is vertex
  operations: add a vertex and join it by an edge to a vertex of G
- Series Parallel graphs (K_4-free)
  prime graph: G is a loop
  operations: subdivide an edge or duplicate an edge.



The tree decomposition of a graph gives a way to apply a polyhedral approach:

- Maximum stable set (Boulala and Uhry 1979, Mahjoub 1988, on SP graphs)
- Maximum cut (Barahona 1983, on graphs not contractible to K_5)

Definitions

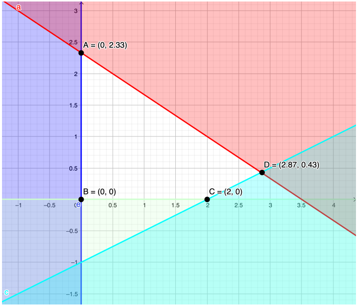
- $H = \{x \in \mathbb{R}^n : a^T x = b\}$ *Hyperplane*, $a \in \mathbb{R}^n$; $b \in \mathbb{R}$.
- $H_+ = \{x \in \mathbb{R}^n : a^T x \geq b\}$ and $H_- = \{x \in \mathbb{R}^n : a^T x \leq b\}$ *half- spaces* such that $H_+ \cup H_- = \mathbb{R}^n$

- *A polyhedron* is the intersection of a finite set of closed half-spaces.

- A non empty bounded polyhedron is called *a polytope* .

- A polytope is the *convex hull* of a finite number of points (*extreme points*).

Example in $\mathbb{R}^2$



In this example the polytope is equal
$Conv(\{A, B, C, D\}) = \{2x + 3y \leq 7; x - 2y \leq 2; x \geq 0; y \geq 0\}$

Optimization problems on a graph

Let $G = (V, E)$ *be a finite graph ;*

$\mathcal{S}$ *is the set of subsets of vertices (or edges) verifying some property* π.

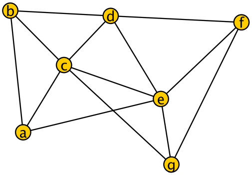
Find a solution $S \in \mathcal{S}$ *such that* $c(S) = \sum_{t \in S} c(t)$ *is maximum or minimum, where* $c(t)$ *is the weight of the vertex (or edge)* t.

Examples for nodes

Given a graph $G = (V, E)$;

Subsets of vertices

- $S \subset V$ is a stable if it induces the subgraph $(S, E(S) = \emptyset)$
- $S \subset V$ is a clique if it induces the complete subgraph $K_{|S|}$
- $S \subset V$ is a dominant if each vertex of $V \setminus S$ is adjacent to a vertex of S
- $S \subset V$ is a node-cutset if G is connected and $V \setminus S$ induces a non connected subgraph.



Stable={a,d,g}; Clique={a,c,e};

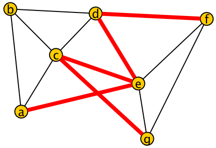
Dominant={b,e};Node-Cutset={c,d,e};

Examples for edges

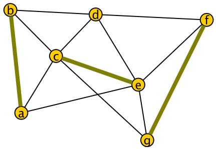
Given a graph $G = (V, E)$;

Subsets of edges

- $S \subset E$ is a cut set if G is connected and $(V, E \setminus S)$ induces a non connected spanning subgraph.
- $S \subset E$ is a matching if no two edges in S share a same endnode.



Cut



Matching

Polyhedral approach

For a graph $G = (V, E)$, we associate a polytope $P = P(G)$ in $\mathbb{R}^V$ (or $\mathbb{R}^E$) as follows:

- Each solution $S \in \mathcal{S}$ is associated to a $\{0, 1\}$ vector $x^S \in \mathbb{R}^V$ (or $\mathbb{R}^E$), the *incidence vector* of S such that:

$$x^S(t) = \begin{cases} 1 & \text{if t is in } S, \\ 0 & \text{otherwise} \end{cases}$$

- $P(G) = Conv(\{x^S : S \in \mathcal{S}\})$
- Try to find a finite set of inequalities (*defining a linear system* of P) such that x^S is a solution of this system iff it belongs to P.
- Then finding the maximum (or minimum) weighted S is a (non integer) linear programming problem subject to the defining system of P.

V. Chvàtal (75) *On certain polytopes associated with graphs*

Let $S(G)$ be the set of all $\{0,1\}$ incidence vectors $(x_u, u \in V)$ of stable sets in $G = (V, E)$ and $P(G)$ the convex hull of $S(G)$.

- Nonnegativity constraints

$$x_u \geq 0, \quad u \in V \tag{1}$$

- Clique constraints

$$\sum_{u \in K} x_u \leq 1, \quad K \in \mathcal{K}(G) \tag{2}$$

where $\mathcal{K}(G)$ is the set of all maximal cliques in G;

- Odd hole constraints

$$\sum_{u \in C} x_u \leq \frac{|C| - 1}{2}, \quad C \in \mathcal{C}(G) \tag{3}$$

where $\mathcal{C}(G)$ is the set of all odd hole of G.

Theorem (V. Chvàtal, 1975)

1. G is perfect iff the inequalities (1) and (2) are a defining linear system of $P(G)$
2. G is h-perfect iff the inequalities (1), (2) and (3) are a defining linear system of $P(G)$

Separations, V. Chvàtal (75)

$G_1 = (V_1, E_1)$; $G_2 = (V_2, E_2)$; $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$
Let the following inequalities be defining linear systems of $P(G_k), k = 1, 2$

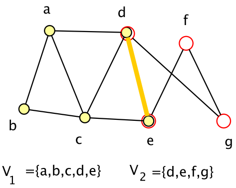
$$x_u \geq 0, \quad u \in V_k \tag{4}$$

$$\sum_{u \in V_k} a_{iu} x_u \leq b_i, \quad i \in J_k \tag{5}$$

Theorem (V. Chvàtal, 1975)

If $G_1 \cap G_2$ is complete then the *union* of ((4),(5)) for $k = 1, 2$, is a defining linear system of $P(G_1 \cup G_2)$.

Example : Separations



$P(G_1 \cup G_2)$

$\{x_a + x_b + x_c \leq 1; x_a + x_c + x_d \leq 1; x_c + x_d + x_e \leq 1\}$

$\{x_d + x_e \leq 1; x_e + x_f \leq 1; x_d + x_g \leq 1; x_f + x_g \leq 1\}$

$x_u \geq 0, u \in \{a, b, c, d, e, f, g\}$

Substitutions,V. Chvàtal (75)

$G_1 = (V_1, E_1)$; $G_2 = (V_2, E_2)$; $V_1 \cap V_2 = \emptyset$
Disjoint union G: Substitute a vertex $v \in V_1$ by G_2 and join each vertex of G_2 to each neighbour of v.
For each $i \in J_1$, set $a^+_{iv} = max\{a_{iv}, 0\}$

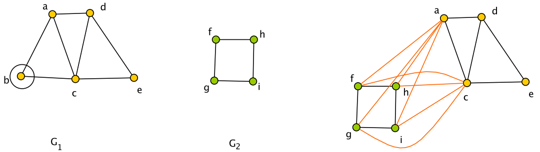
Theorem (V. Chvàtal, 1975)

The following inequalities are defining linear system for $P(G)$

$$x_u \geq 0, \quad u \in V_2 \cup V_1 \setminus \{v\} \tag{6}$$

$$a^+_{iv} \sum_{u \in V_2} (a_{ju} x_u) + b_j \sum_{u \in V_1 \setminus \{v\}} (a_{iu} x_u) \leq b_j b_i, \quad i \in J_1, j \in J_2 \tag{7}$$

Example: Substitutions



G_1 G_2 Disjoint union G

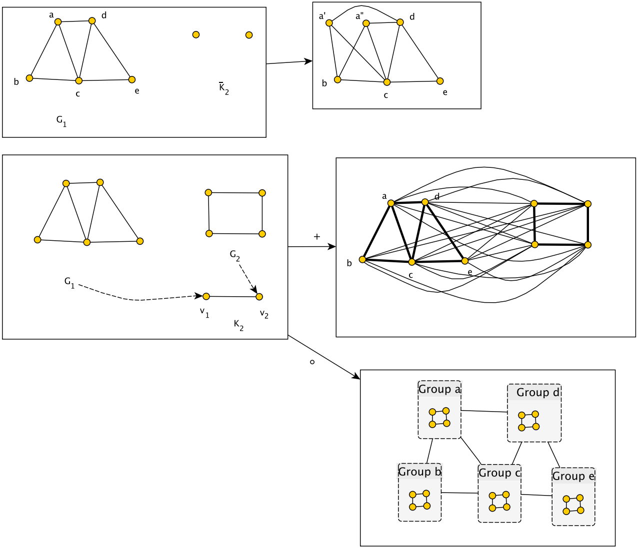
$P(G)$
$\{x_a + x_f + x_g + x_c \leq 1; x_a + x_f + x_h + x_c \leq 1;$
$x_a + x_g + x_i + x_c \leq 1; x_a + x_h + x_i + x_c \leq 1;$
$x_a + x_c + x_d \leq 1; x_c + x_d + x_e \leq 1;$
$x_u \geq 0, u \in \{a, c, d, e, f, g, h, i\}.\}$

More Substitutions

All the following operations are substitution operations which correspond to operations on the polytopes $P(G)$.

- Duplicating v in G_1 $\Leftrightarrow$ substituting $\bar{K}_2$
- A join $G_1 + G_2 \Leftrightarrow$ substitute first G_1 for v_1 and then G_2 for v_2 in the clique K_2 with two vertices v_1 and v_2,.
- A corona $G_1 \circ G_2$ $\Leftrightarrow$ G_2 is substituted for each vertex of G_1

Substitutions Examples



M. Burlet , J. Fonlupt (95) *Polyhedral consequences of the amalgam operation*

In 1995, Burlet and Fonlupt give the amalgam operation $\varnothing$ that starting from 2 graphs G_1 and G_2, builds the graph $G = G_1 \varnothing G_2$ such that :

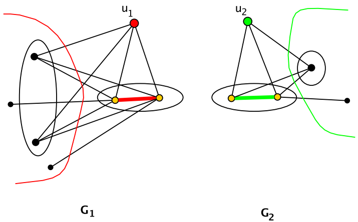
- the description of $P(G)$ is obtained by a system of linear inequalities derived from linear representations of $P(G_1)$ and $P(G_2)$.

- the operation $\varnothing$ preserves perfectness.

Amalgam operation

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ including, for $i = 1, 2$, $u_i \in G_i$ and a clique $C_i \subseteq N_{G_i}(u_i)$ such that

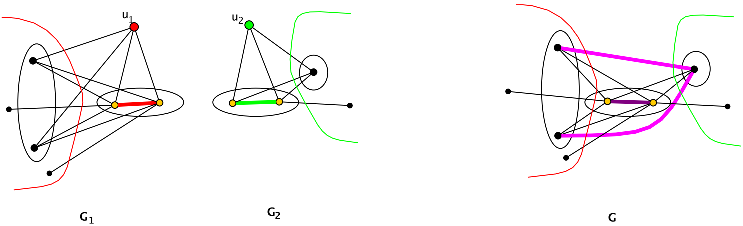
- $|C_1| = |C_2|$
- For $i = 1, 2$, each vertex of C_i is adjacent to each vertex of $N_{G_i}(u_i) \setminus C_i$.
- $N_{G_1}(u_1) \setminus C_1 = \emptyset \Leftrightarrow N_{G_2}(u_2) \setminus C_2 = \emptyset$



Amalgam operation

G is the amalgam of (G_1, u_1, C_1) and (G_2, u_2, C_2)

1. one to one identification of the vertices of C_1 with the vertices of C_2
2. create an edge between every vertex of $N_{G_1}(u_1) \setminus C_1$ and every vertex of $N_{G_2}(u_2) \setminus C_2$.
3. Delete u_1 and u_2.



1985 Cornuéjols and Cunningham :
A polynomial-time algorithm that determines whether or not a given G is the amalgam of two graphs ; If yes then it finds the two graphs.

1994 Burlet and Fonlupt :
- Amalgamation preserves perfectness (They use the result of Chvatal)
-Amalgam generalizes separation and substitution operations :

- separation: $G = (G_1, u_1, C_1) \varnothing (G_2, u_2, C_2)$ where $N_{G_i}(u_i) \setminus C_i = \emptyset$
- disjoint union: the amalgam is $G = (G_1, u_1) \varnothing (G_2', u_2)$ where G_2' is G_2 with the universel vertex u_2.

A few references

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M. Boulala, J.P. Uhry, Polytope des indépendants d'un graphe série-parallèle(1979)
F. Barahona, A.R. Mahjoub, Composition of graphs and polyhedra II: stable sets.(1994)
J. Fonlupt, A.Hadjar, The stable set polytope and some operations on graphs (2002).
G. Stauffer, On the stable set polytope of claw-free graphs (2005).
B. McClosky, I.V. Hicks, Composition of stable set polyhedra (2008).
A. Galluccio, C. Gentile, P. Ventura Gear composition and the stable set polytope (2008).

A glimpse on the k-sum Operation

$G = (V, E)$; $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$
G is a k-sum of G_1 and G_2 if

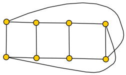
- $V = V_1 \cup V_2$ and $|V_1 \cap V_2| = k$
- $(V_1 \cap V_2, E_1 \cap E_2)$ is a complete subgraph.
- $V_1 \cap V_2$ is a node-cutset

Used in many compositions for polytopes
→ F. Barahona, The max cut problem in graphs non contractible to K_5 (1983)
→-R. Euler and A.R. Mahjoub On a composition of independence systems by circuit identification (1990)

F. Barahona, The max cut problem in graphs non contractible to K_5 (1983)

Wagner's theorem 1937

A graph is K_5-minor-free if and only if it can be built from planar graphs and V_8 by repeated k-sums.



Barahona's theorem 1983

If G is a k-sum of G_1 and G_2 then $P_{cut}(G)$ is obtained by juxtaposing the inequalities which define $P_{cut}(G_1)$ and $P_{cut}(G_2)$ and identifying the variables associated with edges in $G_1 \cap G_2$.

$\rightarrow$ $P_{cut}(G)$ can be obtained for graphs not contractible to K_5.

$\rightarrow$ A polynomial combinatorial algorithm.

Triangle free subgraph Polytope

Let $P_\Delta(G) = Conv\{x^T \in \mathbb{R}^E : (V,T) \text{ is triangle free}\}$. The triangle Free subgraph problem is

$$Max\{wx : x \in P_\Delta(G)\}$$

$x \in P_\Delta(G)$ satisfies

$$\sum_{e\in\Delta} x(e) \quad \leq 2, \Delta \subset E \tag{8}$$

$$0 \leq x(e) \leq 1, \qquad e \in E \tag{9}$$

(8) are the *Triangle inequalities* and (9) are the *Trivial inequalities*.

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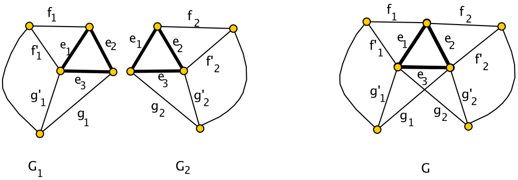
- Euler and Mahjoub (1990): Studied $P_{\Delta}(G)$ in the graphs decomposable by 1- and 2-sums : $P_{\Delta}(G)$= union of $P_{\Delta}(G_1)$ and $P_{\Delta}(G_2)$ by identifying the variables associated to the edges of $E_1 \cap E_2$.
- Bendali, Mahjoub and Mailfert (2002) : Studied $P_{\Delta}(G)$ in graphs decomposable by a special 3-sum.



Mixed constraints

Let $F_k = \{f_k, g_k, e_1, e_2, e_3\}, k = 1, 2$; Given two valid constraints of $P_\Delta(G_1)$ and $P_\Delta(G_2)$

$$\sum_{e \in F_1} a(e)x(e) + \sum_{E_1 \setminus F_1} a(e)x(e) \leq b \qquad (10)$$

$$\sum_{e \in F_2} a(e)x(e) + \sum_{E_2 \setminus F_2} a(e)x(e) \leq b' \qquad (11)$$

we obtain a valid constraints of $P_\Delta(G)$ of the form

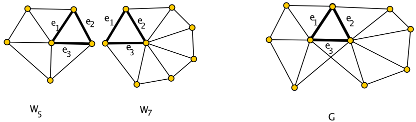
$$\sum_{e \in E_1 \setminus F_1} a(e)x(e) + \sum_{e \in E_2 \setminus F_2} a(e)x(e) + \sum_{F_1 \cup F_2} a(e)x(e) \leq b + b' - 2($$

The wheel W_n and its polytope

A wheel W_n is a cycle on n vertices and a universal node.
If $n = 2k + 1$, from a result of Conforti, Corneil and Mahjoub (1986), $P_{\triangle}(W_n)$ is completely described by the trivial and triangle inequalities together with the inequalities:

$$\sum_{e \in W_{2k+1}} x(e) \quad \leq 3k + 1 \qquad (13)$$

3-sum of two wheels



$P_\Delta(G)$ is completely described by trivial inequalities and

$$\sum_{e\in W_5} x(e) \leq 7, \tag{14}$$

$$\sum_{e\in W_7} x(e) \leq 10, \tag{15}$$

$$\sum_{e\in E} x(e) \leq 15, \tag{16}$$

$$\sum_{e\in \Delta} x(e) \leq 2, \Delta \subset E \tag{17}$$

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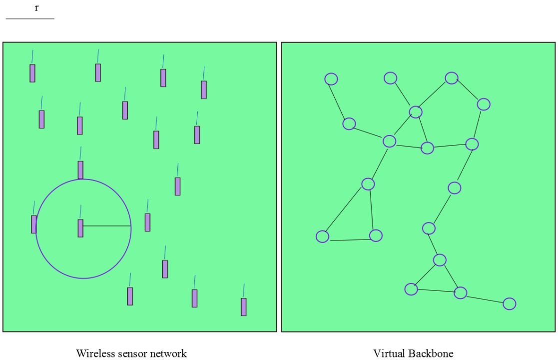
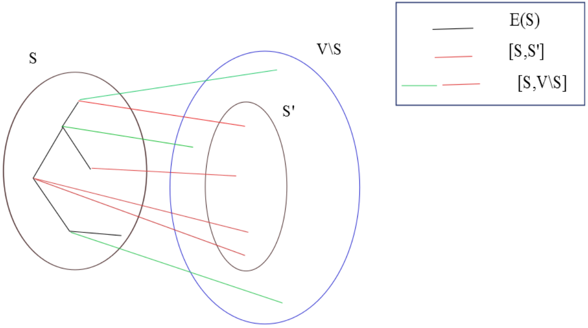


Figure: Communication graph $G = (V, E)$

Definitions

$G = (V, E)$ a graph.

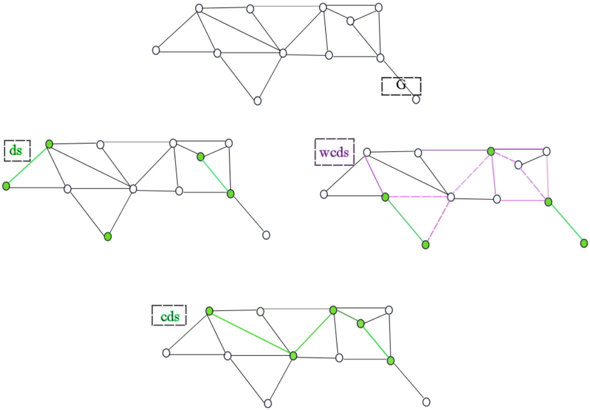
- $S \subset V$, $S' \subset V$ s.t. $S \cap S' = \emptyset$, $[S, S']$ denotes the set of edges with exactly one end in each set.
- $S \subset V$, $E(S)$ edges with both ends in S.



Definitions

$G = (V, E)$ a graph.

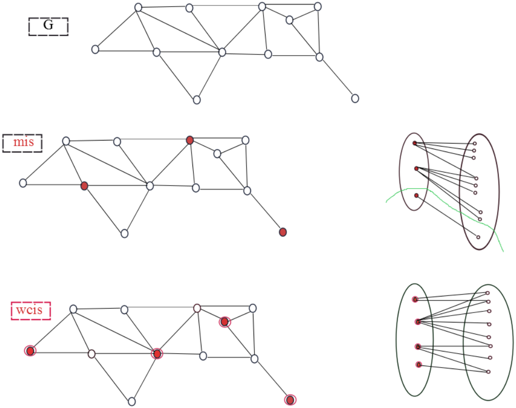
- *Dominating Set (ds)*
  $D \subset V$ is a *ds* if $\forall v \in V \setminus D$, $\exists u \in D$ s.t. $(u, v) \in E$.
- *Connected Dominating Set (cds)*
  A dominating set $D \subset V$ is a *cds* if $G(D) = (D, E(D))$ is connected.
- *Weakly Connected Dominating Set (wcds)*
  A dominating set $D \subset V$ is a *wcds* if $G(D) = (V, E(D) \cup [D, V \setminus D])$ is connected.


G
ds
wcds
cds

Definitions

$G = (V, E)$ a graph.

- *Maximal Independent Set (mis)*
  $S \subset V$ is a *mis* if S is a dominating independent set.
- *Weakly Connected Independent Set (wcis)*
  An independent set $W \subset V$ is a *wcis* if
  $G_W = (V, [W, V \setminus W])$ is connected.


G
mis
wcis

wcis in WSN

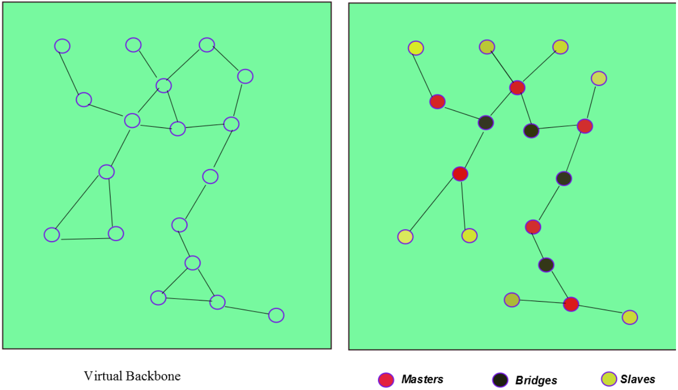


Figure: A wcis = A hierarchy in a wireless sensor network

Minimum weakly connected independent set problem (F. B, J. Mailfert, D. Mameri (2016))

$G = (V, E)$ a connected graph, $W \subset V$ a wcis.
$\{0, 1\}$ vector x^W: the incidence vector of V given by

$$x^W(v) = \begin{cases} 1 & \text{if } v \in W \\ 0 & \text{if } v \in V \setminus W \end{cases}$$

MWCIS problem

$$min\{\sum_{v \in W} x(v) : W \text{ weakly connected independent set of } G\}$$

$$P_{wcis}(G) = Conv\{x^W \in \mathbb{R}^{|V|} : W \subset V, \text{ wcis in } G\}.$$

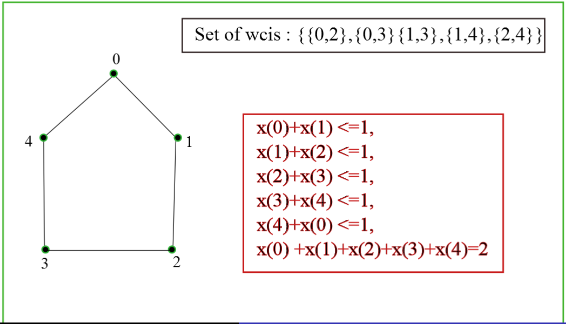
P_{wcis} in Odd Cycles

Theorem

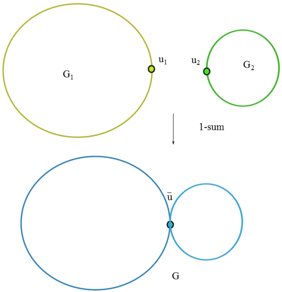
$P_{wcis}(C_{2k+1})$ is completely described by the following system.

$$\sum_{i=0}^{2k} x(i) = k,$$

$$x(i) + x(i+1) \leq 1, \qquad \text{for } i = 0, \ldots, 2k.$$



P_{wcis} in 1-sum



$$(P_{wcis}(G_i)) \begin{cases} \sum_{v \in V_i'} a_j^i(v)x(v) + a_j^i(u_i)x(u_i) \le \alpha_j^i, & j \in J_i \\ x(v) \ge 0, & \forall v \in V_i \end{cases}$$

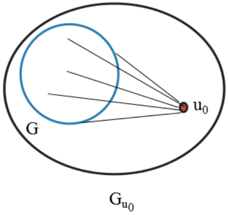
For $i = 1, 2$

Theorem

If $G = (V, E)$ is the 1-sum of G_1 and G_2 obtained by identifying a vertex u_1 and a vertex u_2, then $P_{wcis}(G)$ is given by

$$\begin{cases} \sum_{v \in V_i'} a_j^i(v)x(v) + a_j^i(u_i)x(\bar{u}) \le \alpha_j^i, & j \in J_i, i = 1, 2 \\ x(v) \ge 0, & \forall v \in V_1' \cup V_2' \cup \{\bar{u}\}. \end{cases}$$

P_{wcis} by adding a universal node



The convex hull of maximal independent sets $P_{mis}(G)$ is given by constraints of the form

$$\sum_{u \in V} a_i(u)x(u) \leq \alpha_i, i \in I.$$

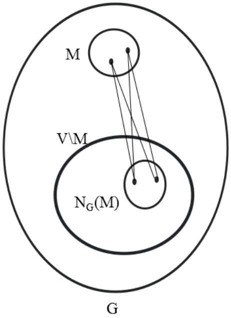
The polytope $P_{wcis}(G_{u_0})$ is described by

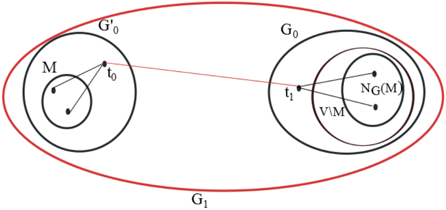
$$\sum_{u \in V} a_i(u)x(u) + \alpha_i x(u_0) \leq \alpha_i, \qquad i \in I,$$

$$x(u_0) \geq 0.$$

P_{wcis} by modules

Let $G = (V, E)$ be a connected graph. A set $M \subseteq V$ of vertices is *a module* of G if for any pair $u, v \in M$, $N(u) \cap (V \setminus M) = N(v) \cap (V \setminus M)$.





$P_{wcis}(G_1)$:

- $P_{mis}(M)$ adding a universal node $t_0 \rightarrow P_{wcis}(G'_0)$;
- $P_{wcis}(G_0)$;
- The 1-sum of (1-sum of G_0 and an edge) and G'_0;

$P_{wcis}(G)$ is obtained from a projection of $P_{wcis}(G_1)$.

Decomposition of graphs and composition of polytopes

Introduction

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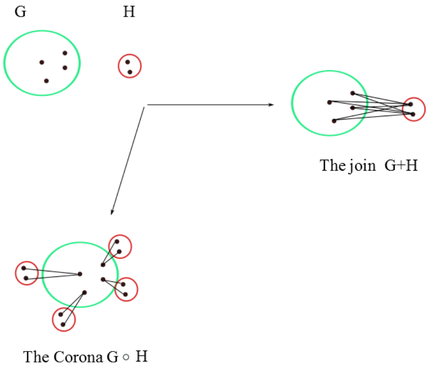
Other compositions

Compositions for the Weakly Connected Independent set polytope

Conclusion

P_{wcis} by Join and Corona Operations (F. B, J. Mailfert (2018))



P_{wcis} by Join and Corona Operations

- In the Join $G + H$:
  H (G) is a module of $G + H$: $P_{mis}(H)$ and $P_{mis}(G) \rightarrow P_{wcis}(G + H)$.
- In the Corona $G \circ H$:
  A universal node v and H : $P_{mis}(H) \rightarrow P_{wcis}(H \cup \{v\})$.
  $P_{wcis}(G \circ H)$ is obtained from 1-sums of G and the copies of $H \cup \{v\}, v \in V(G)$.

Berge's conjecture (1960) → Lovasz (1972) → perfect, h-perfect and compositions (1975-2000)→ Seymour et al. → New operations→ New composition of graphs → New facets and polytopes

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Thank you! ... and join us next June for