Piecewise models for Sequential Convex MINLP technique

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1 Introduction

Our work aims to solve a class of Mixed-Integer Nonlinear Programs (MINLPs) in which we can decompose the non-convex part into sums of univariate non-convex functions. [3] defines these problems and proposes the Sequential Convex Mixed Integer Non-Linear Programming (SC-MINLP) technique. We follow the notation below :

$$\begin{aligned} \min\ & \textstyle\sum_{j\in N} c_j x_j \\ & f_i(x) + \textstyle\sum_{j\in H(i)} g_{ij}(x_j) \le 0, i \in M \\ & l_j \le x_j \le u_j, j \in N \\ & x_j \in \mathbb{Z}, j \in I. \end{aligned}$$

The functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are convex and can be multivariate, while functions $g_{ij} : \mathbb{R} \to \mathbb{R}$ are non-convex univariate. All sets M, N, $I \subseteq N$, and $H(i) \subseteq N$ are finite and l_j and u_j are finite bounds for x_j that appear in g_{ij} functions. The problem is $\mathcal{NP}$-hard and can represent variations of this structure since the notation presented is simplified. For example, the objective function can also have the $f_i(x) + \sum_{j\in H(i)} g_{ij}(x_j)$ form, as is showed in [3] for the Non-linear Knapsack problem. Also, not all variables need necessarily appear in some non-convex terms, and the bounds $l_j \le x_j \le u_j$ are not necessary for this case.

In the SC-MINLP technique, the relaxation of the $g_{ij}(x_j)$ functions is performed as follows : a commercial package computes the $s(ij)+1$ breakpoints $l_j = l^1_{ij} < l^2_{ij} < ... < l^{s(ij)}_{ij} < l^{s(ij)+1}_{ij} = u_j$ from the second derivative of $g_{ij}(x_j)$. Then, we can define whether each subinterval $[l^s_{ij}, l^{s+1}_{ij}]$ is convex $\check{S}_{ij}$ or concave S_{ij}. Convex intervals are left as they are, while a linear relaxation replaces concave intervals. These steps define a convex MINLP, whose continuous relaxation provides valid lower bounds. In [3, 2], the following *Incremental Model* is used to formulate the piecewise-convex functions. However, there are different ways to make this reformulation.

In this work, we compare the three classical different formulations for piecewise problems [1] : the *Incremental Model*, the *Multiple Choice*, and the *Convex-Combination*. For piecewise-linear functions, these models are known to be equivalent, as showed in [1]. However, this is not the case for non-linear piecewise-convex functions, where *Incremental Model* is weaker than others.

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2 Alternative formulations for SC-MINLP

In the *Incremental Model* (IM), each sub-interval $[l_{ij}^s, l_{ij}^{s+1}]$ has a segment load variable x_{ij}^s which assumes value zero unless x_{ij}^{s-1} reaches its maximum value, that is, $x_{ij}^{s+1} > 0$ only if $x_{ij}^s = l_{ij}^{s+1} - l_{ij}^s$. The binary variables y_{ij}^s are defined by the condition that $y_{ij}^s = 1$ if $x_{ij}^s > 0$, and $y_{ij}^s = 0$ otherwise. The variables z_{ij}^s define the contribution of each convex interval to the relaxed function. For $\bar{f}_i = f_i(x) + \sum_{j\in H(i)} g_{ij}(l_{ij}^1) + \sum_{s\in\hat{S}(ij)} \alpha_{ij}^s x_{ij}^s$, we can define :

$$\min \textstyle\sum_{j\in N} c_j x_j \tag{1}$$

$$\bar{f}_i(x) + \textstyle\sum_{j\in H(i)} \sum_{s\in\check{S}(ij)} z_{ij}^s \leq 0 \qquad i \in M \tag{2}$$

$$z_{ij}^s \geq g_{ij}(l_{ij}^s + x_{ij}^s) - g_{ij}(l_{ij}^s) \qquad s \in \check{S}(ij),\ j \in H(i),\ i \in M \tag{3}$$

$$x_j = l_j + \textstyle\sum_{s\in S(ij)} x_{ij}^s \qquad j \in H(i), i \in M \tag{4}$$

$$(l_{ij}^{s+1} - l_{ij}^s) y_{ij}^{s+1} \leq x_{ij}^s \leq (l_{ij}^{s+1} - l_{ij}^s) y_{ij}^s \qquad s \in S(ij),\ j \in H(i),\ i \in M \tag{5}$$

$$y_{ij}^s \in \{0, 1\} \qquad s \in S(ij),\ j \in H(i),\ i \in M \tag{6}$$

$$x_j \in \mathbb{Z} \qquad j \in I \tag{7}$$

The *Multiple Choice* model (MC) introduces an alternative definition of the segment variables. The load variable x_{ij}^s, for each segment s, defines the total load $x_{ij}^s = x_j$ and $y_{ij}^{s+1} = 1$, if x_j lies on the sub-interval $[l_{ij}^s, l_{ij}^{s+1}]$. Otherwise, $x_{ij}^s = y_{ij}^{s+1} = 0$. In this formulation, at most one y_{ij}^{s+1} will equal one. The model is defined with (1), (2), (6), (7), plus :

$$z_{ij}^s \geq g_{ij}(x_{ij}^s) - g_{ij}(0) \qquad s \in \check{S}(ij),\ j \in H(i),\ i \in M \tag{8}$$

$$x_j = \textstyle\sum_{s\in S(ij)} x_{ij}^s \qquad j \in H(i), i \in M \tag{9}$$

$$l_{ij}^s y_{ij}^s \leq x_{ij}^s \leq l_{ij}^{s+1} y_{ij}^s \qquad s \in S(ij),\ j \in H(i),\ i \in M \tag{10}$$

$$\textstyle\sum_{s\in S(ij)} y_{ij}^s = 1 \qquad i \in M,\ j \in H(i) \tag{11}$$

and redefining $\bar{f}_i = f_i(x) + \sum_{j\in H(i)} g_{ij}(0) \sum_{s\in\check{S}(ij)} y_{ij}^s + \sum_{s\in\hat{S}(ij)} (\alpha_{ij}^s x_{ij}^s + (g_{ij}(l_{ij}^s) - \alpha_{ij}^s l_{ij}^s) y_{ij}^s)$.

Finally, the *Convex-Combination* model (CC) is similar to the MC, but replacing x_{ij}^s by defining multipliers μ_{ij}^s and λ_{ij}^s as the weights of these two endpoints. The load and its cost are computed as a convex combination of the load/cost of the two endpoints of the segment. For this formulation, we replace x_{ij}^s by $l_{ij}^s \mu_{ij}^s + l_{ij}^{s+1} \lambda_{ij}^s$ in equations (8) and (9). The replacement is also made in $\bar{f}$, defined for MC. In addition, we have : (1), (2), (6), (7) and $\mu_{ij}^s + \lambda_{ij}^s = y_{ij}^s$, $s \in S(ij),\ j \in H(i),\ i \in M$.

All formulations provide the same integer optimal solution, i.e., they are all equivalent. However, unlike the linear case, the continuous relaxations are *not* equivalent : indeed, we can provide counter-examples where IM provides worse lower bound than the other two. Computational results show that MC and CC always provide equivalent lower bounds, while IM does indeed yield worse in practice ; however, this does not always imply worse CPU times. Future research will comprise a more in-depth computational analysis, as well as the theoretical proof of the equivalence between MC and CC and that both have better relaxation than IM.

Références

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