Minimal winning coalitions and orders of criticality

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1 Introduction

In this paper, we analyze the order of criticality in simple games, under the light of minimal winning coalitions. The order of criticality of a player in a simple game is based on the minimal number of other players that have to leave so that the player in question becomes pivotal [2]. We show that this definition can be formulated referring to the cardinality of the minimal blocking coalitions or minimal hitting sets for the family of minimal winning coalitions ; moreover, the blocking coalitions are related to the winning coalitions of the dual game.

The orders of criticality considered in this paper may be useful when the situation under investigation is unstable. For instance, we may think of a political majority in a parliament where some parties are critical of the first order, i.e. they are able to reject a proposal of the government if they vote against it, while the parties that have a higher order of criticality have this possibility only if they join to other parties ; in a particular unstable situation, it can be useful to assess the effective influence of these parties that may have the chance to destroy the majority.

In this paper, we also investigate the use of the notion of orders of criticality (for the grand coalition) to rank players according to their "power" to block decisions in a simple game. To be more specific, we propose to rank all the players lexicographically accounting the number of coalitions for which they are critical of each order, and we characterize this ranking using four independent axioms. We propose four properties that a solution (i.e., a map that associates to any simple game a total preorder, or ranking, over the individual players) should satisfy. Along the lines of the example of a parliament introduced earlier (where the agents are political parties), the first property we introduce (*players' anonymity*) is a classical anonymity property, saying that the name of the parties should not affect the final ranking. The second property (*dual coalitional anonymity*) considers two distinct parliaments and two parties which belong to the same number of minimal blocking coalitions of same size in the two parliaments : then the two parties should have the same relative power under the two parliaments. The third axiom (*dual monotonicity*) states that if two parties share the same position of a ranking in a given parliament, increasing the number of minimal blocking coalitions containing only one of the two parties should break the tie in favour of the party that now has more possibility to block a decision. Finally, the last property (*independence of higher cardinalities*), is a coherence principle affirming that, once a party is considered strictly more powerful than another one in a parliament, adding coalitions which are minimal, but larger than those already present in the government, should not affect the relative ranking of the two parties. We then show that a solution satisfies these four properties if and only if it is the *criticality-based ranking*, which ranks parties according to a lexicographic comparison of vectors whose components represent the number of times each party is critical of any order among the minimal winning coalitions. So, any party critical of order one is more important than any party critical, at most, of order two, which is, in turn, more important than any party critical, at most, of order three, etc. It is worthy to note that the four properties used to axiomatically characterize the criticality-based ranking are similar, from a technical point of view, to those used in [1] to axiomatically characterize the *lex-cel* social ranking solution in a completely different domain, but they suggest a very different interpretation.

Références

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