A finite ε-convergence algorithm for 0-1 mixed-integer convex two-stage robust optimization with objective uncertainty

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1 Introduction

In this work, we study the wide class of optimization problems where some costs are not known at decision time and the decision flow is modeled as a two-stage process. We show how two-stage robust models for this class of problems can be solved by means of a branch-and-price algorithm where one may branch on continuous values so as to tighten the optimality gap. Our approach generalizes a recent result from the literature which addressed the linear case and was only applicable in presence of linking constraints involving binary variables ([1]), and extends the associated results to problems with convex constraints and general mixed-integer linking constraints. The convergence of the method is proven to ε-optimality.

2 Problem statement

More formally, we consider uncertain optimization problems where a two-stage decision flow is required. Mixed-integer variable $\boldsymbol{x}$ are used to model the decisions to be taken here and now (i.e., before the realization of the uncertainty), whereas mixed-integer variable $\boldsymbol{y}$ model the decisions to be taken in a wait-and-see phase (i.e., once the uncertainty is revealed). Uncertainty is modeled by continuous variables $\boldsymbol{\xi}$ that belong to a bounded convex set Ξ.

Our problem of interest then reads :

$$\min_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{\xi}\in\Xi} \min_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{\xi},\boldsymbol{y}) \tag{2SRO-P}$$

Here, $\mathcal{X}$ denotes the first-stage feasible set while, for any $\bar{\boldsymbol{x}}\in\mathcal{X}$, $\mathcal{Y}(\bar{\boldsymbol{x}})$ denotes the set of admissible second-stage decisions. The continuous relaxation of $\mathcal{X}$ and $\mathcal{Y}(\boldsymbol{x}), \forall \boldsymbol{x}\in\mathcal{X}$ is assumed to be convex, closed and bounded. In particular, denoting by X^* the set of optimal first-stage solutions of (2SRO-P), we assume that one is able to find a hyper rectangle $[\boldsymbol{l},\boldsymbol{u}]$ such that $X^*\cap[\boldsymbol{l},\boldsymbol{u}]\neq\emptyset$. Function f is assumed to fulfill mild assumptions, ensuring the validity of the following reformulation of (2SRO-P) :

$$\min_{\boldsymbol{x}\in\mathcal{X}\cap[\boldsymbol{l},\boldsymbol{u}]} \max_{\boldsymbol{\xi}\in\Xi} \min_{(\boldsymbol{t},\boldsymbol{y})\in\mathcal{Y}'(\boldsymbol{x})} \sum_{i\in Q} w_i(\boldsymbol{\xi}) t_i \tag{2SRO-R}$$

where $\mathcal{Y}(\boldsymbol{x}) = \text{proj}_{\boldsymbol{y}}(\mathcal{Y}'(\boldsymbol{x}))$ and the continuous relaxation of $\mathcal{Y}(\boldsymbol{x})$ is convex, closed and bounded for all $\boldsymbol{x}$ taken in the continuous relaxation of $\mathcal{X}$ and functions $(w_i)_{i\in Q}$ are concave functions. In what remains of this document, set I will denote the set of indices for variables $\boldsymbol{x}$ while set $I_B\subseteq I$ will comprise indices i for which x_i is required to be integer (i.e., $\boldsymbol{x}\in\mathbb{R}^{|I\setminus I_B|}\times\{0,1\}^{|I_B|}$).

3 Main contribution

Our main contribution is to propose a novel solution approach for problem (2SRO-P) by means of a branch-and-price algorithm where some of the first-stage continuous variables may be selected for branching throughout the execution. Our algorithm relies on the following two proposition.

Proposition 1 (Lower-bounding problem)**.** *The following optimization problem's optimal objective value is always a lower bound on the the optimal objective value of* (2SRO-P) *:*

$$\begin{aligned} \underset{\boldsymbol{x},\boldsymbol{y},(\boldsymbol{v}^i)_{i\in Q},\boldsymbol{\xi}}{\text{minimize}} \quad & \delta^*(\boldsymbol{\xi}|\Xi) - \sum_{i\in Q} t_i w_{i*}\left(\frac{\boldsymbol{v}^i}{t_i}\right) \\ \text{subject to} \quad & \boldsymbol{x} \in \mathcal{X} \cap [\boldsymbol{l}, \boldsymbol{u}] \\ & (\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}) \in \text{conv}\,(S) \\ & \sum_{i\in Q} \boldsymbol{v}^i = \boldsymbol{\xi} \\ & \boldsymbol{v}^i \in \mathbb{R}^{|U|} \qquad \forall i \in Q \\ & \boldsymbol{\xi} \in \mathbb{R}^{|U|} \end{aligned} \tag{P}$$

where $\delta^*(\cdot|X)$ *denotes the convex conjugate of the indicator function of set* X*,* w_{i*} *denotes the concave conjugate of function* w_i *for* $i \in Q$ *and* conv (S) *is the convex hull of the following set :*

$$S = \left\{ (\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}) : \begin{array}{ll} l_j \leq x_j \leq u_j & \forall j \in I \\ x_j \in \{0,1\} & \forall j \in I_B \\ (\boldsymbol{t}, \boldsymbol{y}) \in \mathcal{Y}'(\boldsymbol{x}) & \end{array} \right\} \tag{1}$$

Proposition 2 (Tightness condition)**.** *Denoting by* $v(\cdot)$ *the optimal objective value of problem* $\cdot$*, the following implication holds :*

$$X^* \cap \text{vert}\,([\boldsymbol{l}, \boldsymbol{u}]) \neq \emptyset \Rightarrow v(\text{P}) = v(\text{2SRO-P})$$

where vert $([\boldsymbol{l}, \boldsymbol{u}])$ *denotes the set of extreme points of* $[\boldsymbol{l}, \boldsymbol{u}]$*.*

These two propositions allow us to derive a spatial branch-and-bound algorithm where only the first-stage variables are selected for branching until the tightness condition from proposition (2) is reached. The convex hull of S is dynamically built as an ever increasing polyhedral approximation by means of nonlinear column generation. The obtained method is proven to converge finitely for a given nonzero precision.

We then apply our new methods to two problems. The first one is a variant of the Capacity Facility Location problem where the unitary travel costs are not known at decision time. The opening of facilities must be decided before knowing the exact travel costs while the underlying routing problem is delayed to the second stage. Note that each arc between an opened facility and a customer is also associated to a fixed activation cost one has to pay when using the arc. The uncertain travel costs are assumed to follow a normal probability distribution and the uncertainty set is therefore modeled as an ellipsoidal set. The second problem is a two-stage capital budgeting problem where a budget may be invested in a set of projects with uncertain profits. The profits are computed according to dependent or independent unknown risk factors. The decision maker may choose to invest here and now or after having observed the risk factors. However, early investments enjoy a first-mover advantage whereas a postponed investment only generates a fraction of the profit. We also study the impact of possible loans in the investment plan. For both problems we report encouraging computational results on a large benchmarks derived from instances in the literature.

Références

[1] Ayşe N Arslan and Boris Detienne. Decomposition-based approaches for a class of two-stage robust binary optimization problems. *INFORMS Journal on Computing*, 2021.