Diameter in linear time for constant-dimension median graphs

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1 Contribution

We study here one of the most fundamental problems in algorithmic graph theory related to distances : the *diameter*. Given an n-vertex undirected graph $G = (V, E)$, the diameter is the maximum distance $d(u, v)$, $u, v \in V$, where $d(u, v)$ is the length of shortest (u, v)-paths. It is a basic parameter used to apprehend the structure of a graph.

In this paper, we propose a linear-time algorithm computing the diameter of constant-dimension median graphs. Median graphs form the class of graphs which is certainly the most studied in metric graph theory. They are the graphs such that any triplet of vertices has a unique median. Put formally, given $x, y, z \in V$, there is a unique vertex $m(x, y, z)$ lying at the same time on some shortest (x, y), (y, z), and (z, x)-paths. Said differently, $m(x, y, z)$ is the unique vertex being metrically between x and y, y and z, z and x. Median graphs are *partial cubes*, *i.e.* isometric subgraphs of hypercubes. However, partial cubes are not necessarily median. Furthermore, median graphs are triangle-free, bipartite and do not contain induced $K_{2,3}$. The dimension d of a median graph is the dimension of its largest induced hypercube. This parameter is upper-bounded by $\lfloor \log n \rfloor$.

To the best of our knowledge, there is no subquadratic algorithm for the diameter on median graphs. Bénéteau *et al.* [1] and Ducoffe [2] recently formulated this open question. There exist efficient algorithms for other metric parameters on median graphs : the median set and the Wiener index can be determined in time $O(|E|)$ [1]. Furthermore, subquadratic algorithms have been proposed for the recognition of median graphs [3]. Our contribution follows.

Théorème 1 *There exists a combinatorial algorithm for computing the diameter in time $O^*(2^{d(\log(d)+1)}n)$ on median graphs.*

Notation O^* neglects polynomials of d, which are also poly-logarithmic factors of n. Let Q_k be the hypercube of dimension k. A consequence of this theorem is that, for any $d = O(1)$, the diameter can be determined in linear time $O(n)$ on Q_d-free median graphs. For example, this is the case for cube-free median graphs ($d \leq 2$). Moreover, as Q_4 is not planar, planar median graphs are Q_4-free, so our algorithm is linear for this family of graphs.

2 Overview of the algorithm

A notion is essential to apprehend the structure of median graphs : Θ-classes. We say that the edges uv and xy of a median graph G are in relation Θ_0 if they form a square $uvyx$, where uv and xy are opposite edges. Then, Θ refers to the reflexive and transitive closure of relation Θ_0. The classes of the equivalence relation Θ are denoted by $E_1, \ldots, E_q$. Each Θ-class E_i is a matching cutset. Figure 1 gives examples of Θ-classes. We say two Θ-classes E_i and E_j are *orthogonal* if there is a square $uvyx$ formed by edges of both E_i and E_j. In Figure 1, E_1 and E_3 are orthogonal for example, while E_1 and E_2 are not.

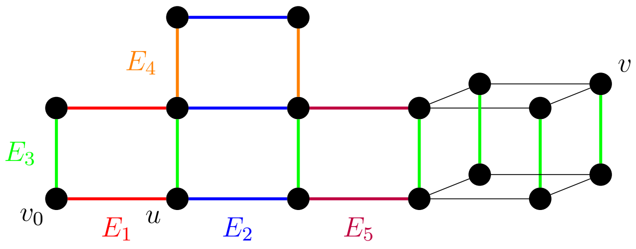

FIG. 1 – A median graph G and five of its classes E_i, $1 \leq i \leq 5$.

Hypercubes are omnipresent in median graphs. However, their cardinality can be upper-bounded by $2^d n$, which is linear for constant dimension. There is a strong relationship between hypercubes and Θ-classes. Indeed, given an hypercube Q of dimension $k \leq d$, each of its edges belong to one of k Θ-classes which are pairwise orthogonal. Such set of Θ-classes is called a Pairwise Orthogonal Family (POF). Our algorithm proceeds in three successive steps.

— **Step 1 : enumeration of hypercubes.** A standard BFS allows us to list the hypercubes of G. Starting from an arbitrary basepoint v_0, for each vertex v, we collect all subsets of incoming edges. There is a bijection between each of these subsets and the hypercubes of G. This step takes time $O^*(2^d n)$.

— **Step 2 : labeling of ladder sets.** The *ladder set* is a notion defined for any pair of vertices u, v aligned with v_0, *i.e.* $d(v_0, v) = d(v_0, u) + d(u, v)$. The ladder set $L_{u,v}$ is the set of Θ-classes incident to u which are on shortest (u, v)-paths. We can prove that $L_{u,v}$ is a POF. In Figure 1, $L_{u,v} = \{E_2, E_3\}$. In this step, we compute labelings $\varphi(u, L)$ for each vertex u and POF L outgoing from u. Integer $\varphi(u, L)$ gives the maximum distance of a shortest path starting from u such that its first edges belong exactly to Θ-classes L. To determine all values $\varphi(u, L)$, we proceed inductively. We observe that there exists a formula between the POFs ingoing into a vertex u and the POF outgoing from it. The running time obtained is $O^*(2^{2d} n)$.

— **Step 3 : maximum-weighted pair of disjoint ladder sets**. Let (u, v) be an arbitrary pair of vertices. There is a unique triplet made up of the median $m = m(u, v, v_0)$ and ladder sets $L_{m,u}$ and $L_{m,v}$ which are disjoint. Determining the diameter is equivalent to finding the pair of labels $\varphi(m, L), \varphi(m, L^*)$ such that $L \cap L^* = \emptyset$ and $\varphi(m, L) + \varphi(m, L^*)$ is maximum. We determine this maximum-weighted pair with a bounded tree search in a tree with at most $d!$ nodes. This step runs in time $O^*(2^{d \log d} 2^d n)$.

As the most expensive step is the third one, the total running time of the algorithm is slightly super-exponential.

3 Perspectives

The algorithm can be naturally extended to compute all eccentricities within the same complexity. We are currently working on the design of a subquadratic algorithm on median graphs where the dimension is not necessarily bounded. The existence of a linear-time algorithm seems to be compromise, as the distance VC-dimension of a median graph is unbounded.

Références

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