Complexity of the Multilevel Critical Node Problem

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Mots-clés : *Trilevel optimization, Polynomial hierarchy, Network interdiction,* Σ_3^p*-completeness.*

1 Multilevel Critical Node

Graphs are powerful mathematical structures that enable us to model real-world networks. The problem of breaking the connectivity of a graph has been extensively studied in combinatorial optimization since it can serve to measure the robustness of a network to disruptions. In this work, we will focus on the Multilevel Critical Node problem (MCN) [1]. Let $G = (V, A)$ be a graph with a set V of vertices and a set A of arcs. In MCN there are two players, designated by defender and attacker, whose individual strategies are given by a selection of subsets of V. The game goes as follows : first, the defender selects a subset of vertices $D \subseteq V$ to *vaccinate* subject to a budget limit Ω and a cost$\{\hat{c}_v\}_{v\in V}$; second, the attacker observes the vaccination strategy, and selects a subset of vertices $I \subseteq V \setminus D$ to (directly) *infect* subject to a budget limit Φ and a cost$\{h_v\}_{v\in V}$; and third, the defender observes the infection strategy, and selects a subset of vertices $P \subseteq V \setminus I$ to *protect* subject to a budget limit Λ and a cost$\{c_v\}_{v\in V}$. A directly or indirectly infected vertex v propagates the infection to a vertex u, if $(v, u) \in A$ and u is neither a vaccinated nor a protected vertex. The goal of the defender is to maximize the benefit b_v of saved vertices (i.e., not infected), while the attacker aims to minimize it. We assume that all parameters of the problem are non-negative integers. The game description can be succinctly given by the following mixed integer trilevel program :

$$(MCN) \quad \max_{\substack{z\in\{0,1\}^{|V|} \\ \sum_{v\in V} \hat{c}_v z_v \leq \Omega}} \quad \min_{\substack{y\in\{0,1\}^{|V|} \\ \sum_{v\in V} h_v y_v \leq \Phi}} \quad \max_{\substack{x\in\{0,1\}^{|V|} \\ \alpha\in[0,1]^{|V|}}} \sum_{v\in V} b_v \alpha_v$$

$$s.t. \quad \sum_{v\in V} c_v x_v \leq \Lambda$$

$$\alpha_v \leq 1 + z_v - y_v, \qquad \forall v \in V \tag{1a}$$

$$\alpha_v \leq \alpha_u + x_v + z_v, \quad \forall\ (u, v) \in A, \tag{1b}$$

where z, y, x and α are decision vectors whose coordinates are z_v, y_v, x_v and α_v for each $v \in V$. In this optimization model, z, y and x reflect the set of vaccinated vertices $D = \{v \in V : z_v = 1\}$, directly infected vertices $I = \{v \in V : y_v = 1\}$ and protected vertices $P = \{v \in V : x_v = 1\}$, respectively. Finally, α mimics the propagation of the infection among the vertices in V, through Constraints (1a) and (1b), and it is necessarily binary due to the maximization in the last level (protection). Concretely, $\alpha_v = 1$ means that vertex v is saved and $\alpha_v = 0$ means that vertex v is infected. In multilevel optimization, the first stage (in MCN, the vaccination stage) is called the upper level or first level, the second stage is called the second level, and so on, with the last stage being also designated by lower level. See [1] for

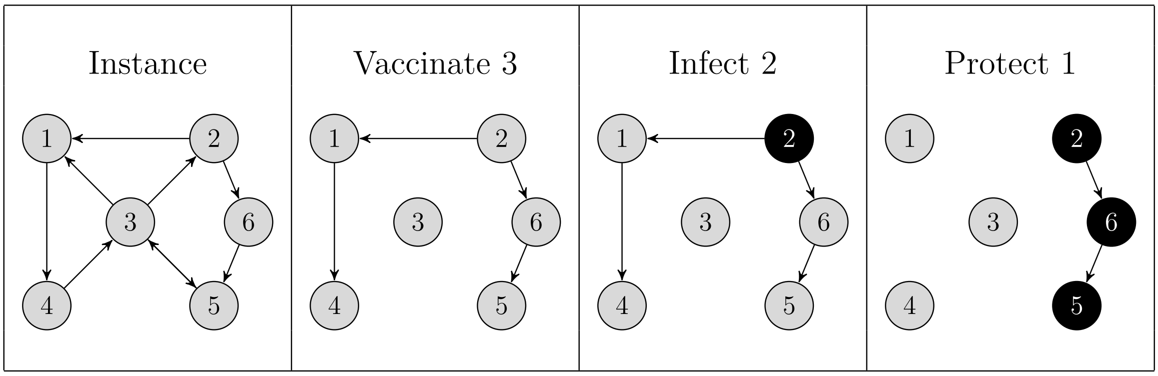

FIG. 1 – Example of an MCN game with unitary costs and benefits, and budgets $\Omega = \Phi = \Lambda = 1$. We removed the vaccinated and protected vertices as an infection cannot pass through them. Vertices $\{1, 3, 4\}$ are saved and $\{2, 6, 5\}$ are infected.

further details on this mathematical programming formulation and Figure 1 for an illustration of the game.

2 Contributions

We have investigated the subgames *(i)* Protect, where given D and I, the defender seeks the optimal protection strategy, *(ii)* Attack, where given D and no protection budget, the attacker determines the optimal infection strategy, *(iii)* Attack-Protect, where given D, the attacker computes the optimal infection strategy, and *(iv)* Vaccination-Attack, where given no budget for protection, the defender finds the optimal vaccination strategy. Note that in a multilevel optimization problem, like the MCN, the ultimate goal is to find the optimal first level decision. Hence, if for example in the MCN, we had always $\Omega = |V|$, then we would know directly that all vertices are saved, even if the attack problem is theoretically intractable. This supports the interest of understanding the individual complexity of each subgame of MCN.

In the presentation we will provide reductions from the Knapsack Interdiction Problem (KIP) [2] to prove that the decision version of both bilevel subgames Attack-Protect and Vaccination-Attack with arbitrary weights are Σ_2^p-complete (where $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$), i.e. are complete for the second level of the Polynomial Hierarchy. These results will then be generalized to demonstrate the Σ_3^p-completeness (where $\Sigma_3^p = (\Sigma_2^p)^{\mathrm{NP}}$) of the decision version of the full trilevel MCN with arbitrary weights using an extension of the KIP to three decision levels. The Σ_3^p-completeness of the trilevel KIP is itself derived from the 3-Alternating Quantified Satisfiability Problem. We therefore provide the first reductions to prove the Σ_3^p-completeness for the decision versions of trilevel interdiction problems and add two problems to the very limited list of Σ_3^p-complete problems. These results shed light on the practical difficulties dealt in [1].

We will also provide a number of complexity results of Protect by exploring graph classes where it becomes polynomially solvable.

Références

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